Approximately achieving the feedback interference channel capacity with point-to-point codes

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Abstract

Superposition codes with rate-splitting have been used for all approximately optimal strategies for the interference channel, with and without feedback. As rate-splitting requires fore-knowledge of channel parameters or statistics, in this paper we explore schemes for the interference channel (with feedback) that do not use superposition or rate-splitting. We demonstrate that point-to-point codes designed for inter-symbol-interference channels, along with time-sharing can approximately achieve the entire rate region of the interference channel with feedback. We show that such a scheme also approximately achieves the rate-region for the interference channel with fading, for a large class of fading distributions.

I. Introduction

Rate splitting is used for all interference channel (IC) strategies that achieve the approximate capacity region, including feedback and non-feedback cases [4], [1], [3]. Such strategies require channel knowledge at the transmitters, as well as joint decoding at the receivers. In this paper we address the question whether we can approximately achieve the capacity region of the Gaussian IC using feedback and point-to-point codes, without using rate splitting.

The approximate capacity of the Gaussian interference channel was characterized in [1] to within one bit, using a rate-splitting scheme and a specific power allocation that allows part of the interference to be received at noise level. The extension of this scheme to the feedback IC was done in [4], where the capacity region was approximated to within 2 bits. Along similar lines, the work in [5] studied the interference channel with rate-limited feedback, and using a scheme that uses superposition coding and lattice codes, the sum capacity was approximated to within a constant number of bits. Fading interference channels were considered in [3], and it was shown that rate-splitting based on the statistics of the channels, as opposed to particular realizations, is approximately optimal for a large class of fading channels. In [6], it was shown that in some regimes of channel parameters the sum capacity could be approximately achieved for the two-user Gaussian IC with successive decoding as opposed to joint decoding. However, the scheme therein still resorted to superposition coding at the transmitters.

In contrast to the strategies in these works, we devise a strategy that does not make use of rate splitting, superposition coding or joint decoding for the feedback IC, and demonstrate that it achieves the entire rate region for symmetric Gaussian interference channels to within a constant gap. Our scheme only uses point-to-point codes, and a feedback scheme based on amplify-and-forward relaying, similar to the one proposed in [4]. Through amplify-and-forward relaying of the feedback signal, the scheme effectively induces a 2-tap inter-symbol-interference (ISI) channel for one of the users and a point-to-point feedback channel for the other user. The work in [4] had similarly shown that an amplify-and-forward based feedback scheme can achieve the symmetric rate point, without using rate splitting. Our work can be considered as an extension to this scheme, which enables us to approximately achieve the entire capacity region of the feedback IC.

We show that our scheme can be extended to a large class of fading interference channels with symmetric channel statistics, without having the instantaneous channel knowledge at the transmitters, using techniques derived in [3]. We can also extend our scheme to rate-limited feedback IC, which approximates the sum capacity provided that the feedback link capacity is greater than $\log(1+INR)$, where INR is the interference-to-noise ratio.

Since the schemes use point-to-point codes and do not use rate-splitting, it could have applicability in scenarios where even the channel statistics could be unknown. We believe that this is a step towards enabling implementable strategies for fading interference channels. The scheme can also potentially enable transmission in interference channel in a rateless fashion, by sending information until the decoding is successful.

The paper is organized as follows. In section II we set up the system model and explain the notation used. In section III we present our scheme and state the main results and in section IV we provide the analysis for the scheme.

II. System Model and Notation

We consider the symmetric two-user static Gaussian interference channel (IC), where the channel inputs (X_1, X_2) at the transmitters 1 and 2 are mapped the outputs (Y_1, Y_2) at receivers 1 and 2 by the equations

$$Y_1(t) = g_d X_1(t) + g_c X_2(t) + Z_1(t)$$
$$Y_2(t) = g_c X_1(t) + g_d X_2(t) + Z_2(t),$$

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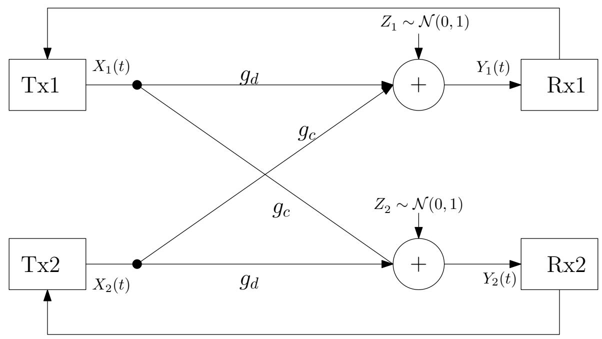

Figure 1. Symmetric Gaussian IC with feedback

where $g_c, g_d \in \mathbb{C}$ are complex channel gains[1], and $Z_i(t) \sim \mathcal{CN}(0,1), i = 1,2$, are circularly symmetric complex white Gaussian noise processes. We assume average power constraint $\frac{1}{n}\sum_{t=1}^{n} |X_i(t)|^2 \leq 1, i = 1,2$, at the transmitters, and assume Txi has a message W_i intended for Rxi for $i = 1,2$, and W_1 and W_2 are independent. At the end of every time slot, Rxi provides channel output feedback to Txi through orthogonal links, denoted by $\hat{Y}_i(t)$, and thus $X_i(t)$ is a function of $(W_i, \hat{Y}_i^{t-1})$. In this paper, we will focus on two different feedback models, the *perfect feedback* model, where $\hat{Y}_i = Y_i$, and the *rate-limited feedback* model, where $\hat{Y}_i^N = f(Y_i^N)$ for some function f and a block length of N, subject to the constraint that $H(\hat{Y}_i^N) \leq NC_{FB}$.

The symmetric two-user fading interference channel similarly consists of the input-output mapping

$$Y_1(t) = g_{11}(t)X_1(t) + g_{12}(t)X_2(t) + Z_1(t)$$
$$Y_2(t) = g_{21}(t)X_1(t) + g_{22}(t)X_2(t) + Z_2(t)$$

where the channel gains $g_{ij}(t), (i,j) \in \{1,2\}^2$ are i.i.d. complex random processes. The $g_{ij}(t)$ processes for any fixed (i,j) are i.i.d across time, and the realizations for different (i,j) are independent. The symmetry assumption implies that g_{11} and g_{22} have the same distribution, *i.e*, $g_{11} \sim g_{22} \sim g_d$, and g_{12} and g_{21} have the same distribution, *i.e.*, $g_{12} \sim g_{21} \sim g_c$. We define $\mu_d := \mathbf{E}\left[|g_d|^2\right]$, and $\mu_c := \mathbf{E}\left[|g_c|^2\right]$. The feedback models considered are similar to those for the static IC.

For schemes involving multiple blocks (phases) we use the notation $X_j^{(i)N}$, where j is the user index, i is the block (phase) index and N is the number of symbols per block. Also $X_2^{(i)}(k)$ indicates the k^{th} symbol in the i^{th} block (phase). We use $\log(\cdot)$ to denote logarithm to the base 2, and define $\log^+(\cdot) := \max(\log(\cdot), 0)$.

III. The scheme and main results

A. Description of the scheme

In this section, we present a scheme for the static IC that only uses point-to-point codes, and no rate splitting. The extension to the fading IC will be presented later.

We consider n transmission phases, each phase having a block length of N. Generate 2^{nNR_1} codewords $\left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$, using i.i.d $\mathcal{CN}(0,1)$ distribution. User 1 encodes its message $W_1 \in \left\{1, \ldots, 2^{nNR_1}\right\}$ onto the corresponding codeword $\left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$. For user 2, generate 2^{nNR_2} codewords X_2^N, and let it encode its message $W_2 \in \left\{1, \ldots, 2^{nNR_2}\right\}$ onto $X_2^{(1)N} = X_2^N$. Note that for user 2, the coding block length is N, whereas our definition of R_2 assumes a transmission duration of nN. This is because, as will become clearer in what follows, Tx2 sends fresh information only during the first block, and uses the subsequent blocks to refine its signal through feedback and amplify-and-forward relaying. We will still denote the vector transmitted by Tx2 at block $i > 1$ with $X_2^{(i)N}$. For clarity of presentation, in this section we will focus on the perfect feedback model. The scheme can be easily modified for the rate-limited feedback with sufficient rate, as will be shown in Section IV.

Tx1 sends $X_1^{(i)N}$ in phase $i \in \{1, \ldots, n\}$. Tx2 sends $X_2^{(1)N} = X_2^N$ in phase 1. For the remaining phases $i > 1$, Tx2 receives

$$Y_2^{(i-1)N} = g_d X_2^{(i-1)N} + g_c X_1^{(i-1)N} + Z_2^{(i-1)N}$$

from feedback. It can remove $g_d X_2^{(i-1)N}$ from $Y_2^{(i-1)N}$ to obtain $g_c X_1^{(i-1)N} + Z_2^{(i-1)N}$. In phase i, Tx2 sends the resulting interference-plus-noise, normalized to meet the power constraint:

$$X_2^{(i)N} = \frac{g_c X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1 + |g_c|^2}}$$

[1]Without loss of generality we assume $|g_c|^2 > 1$, otherwise treating interference as noise provides an approximately optimal scheme.

In phase $i > 1$, Rx2 receives

$$Y_2^{(i)N} = g_d \left(\frac{g_c X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+|g_c|^2}} \right) + g_c X_1^{(i)N} + Z_2^{(i)N}$$

and feeds it back to Tx2 for the next phase. The transmission scheme is summarized in Table I.

Note that with this scheme, for phase $i > 1$, the Tx1 observes a block inter-symbol interference (ISI) channel since its effective channel becomes

$$Y_1^{(i)N} = g_d X_1^{(i)N} + \left(\frac{(g_c)^2}{\sqrt{1+|g_c|^2}} \right) X_1^{(i-1)N} + \tilde{Z}_1^{(i)N}$$

where $\tilde{Z}_1^{(i)N} = Z_1^{(i)N} + \frac{g_c Z_2^{(i-1)N}}{\sqrt{1+|g_c|^2}}$. Note that Tx1-Rx1 pair can employ an OFDM-based scheme to convert the resulting ISI channel to a set of parallel non-interfering channels in the frequency domain (see Appendix B for details).

Table I
n-PHASE SCHEME FOR SYMMETRIC GAUSSIAN IC WITH PERFECT FEEDBACK

User	Phase 1	Phase 2	.	.	Phase n
1	$X_1^{(1)N}$	$X_1^{(2)N}$	.	.	$X_1^{(n)N}$
2	$X_2^{(1)N}$	$\frac{g_c X_1^{(1)N} + Z_2^{(1)N}}{\sqrt{1+\|g_c\|^2}}$	.	.	$\frac{g_c X_1^{(n-1)N} + Z_2^{(n-1)N}}{\sqrt{1+\|g_c\|^2}}$

At the end of n blocks, Rx1 collects $\mathbf{Y_1}^N = \left(Y_1^{(1)N}, \ldots, Y_1^{(n)N}\right)$ and decodes by finding W_1 such that $\left(\mathbf{X_1}^N (W_1), \mathbf{Y_1}^N\right)$ is jointly typical, where $\mathbf{X_1}^N = \left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$. At Rx2, channel outputs over n phases can be combined with appropriate scaling so that the interference-plus-noise at phases $\{1, \ldots, n-1\}$ are successively cancelled, *i.e.*, an effective point-to-point channel can be generated through $\tilde{Y}_2^N = \sum_{i=1}^n \left(\frac{-g_d}{\sqrt{1+|g_c|^2}} \right)^{n-i} Y_2^{(i)N}$ (see proof for details). Note that this can be viewed as a block version of the Schalkwijk-Kailath scheme [2] (and the references therein). Given the effective channel $\tilde{Y}_2^N$, the receiver can simply use point-to-point typicality decoding to recover W_2, treating the interference in phase n as noise.

B. Main results

Proposition 1. *The scheme described in Subsection III-A achieves the rate point*

$$(R_1, R_2) = \left(\log\left(1 + |g_c|^2 + |g_d|^2\right) - 2, \log^+ \left[\frac{|g_d|^2}{1+|g_c|^2} \right] \right)$$

for the symmetric Gaussian IC with perfect feedback.

Since the rate point in Proposition 1 is within a constant number of bits of a corner point of the capacity region of the symmetric feedback IC, and the feedback IC capaciy region has only two non-trivial corner points (see Figure 2), the following theorem follows (with detailed calculations in Appendix C).

Theorem 2. *The scheme described in Subsection III-A, combined with switching the roles of users* 1 *and* 2 *and time-sharing achieves the capacity region of symmetric feedback IC within 2 bits.*

Note that our scheme does not make use of superposition coding or joint decoding, but only uses point-to-point codes. From the perspective of Rx1, the forwarding of feedback at Tx2 simply converts its channel into a point-to-point ISI channel, with the exception of the first block. On the other hand, from the perspective of Rx2, interference is treated as noise which is refined through feedback, which simply converts its own channel into a point-to-point channel with feedback.

Our scheme can be easily extended to the rate-limited setting to obtain the following theorem.

Theorem 3. *For the symmetric Gaussian IC with rate limited feedback, if* $C_{FB} > \log\left(1 + |g_c|^2\right)$ *and* $|g_c|^2 > 1$, *then the rate pair* $(R_1, R_2) =$

$$\left(\log\left(1 + |g_c|^2 + |g_d|^2\right) - 3, \log^+ \left[\frac{|g_d|^2}{3+|g_c|^2} \right] \right)$$

is achievable by the scheme described in Subsection III-A[2].

[2]For rate-limited case, one only needs to modify the scheme such that instead of Rx2 sending complete output and Tx2 subtracting its own signal, Rx2 performs Wyner-Ziv quantization-and-binning of the channel output, which effectively provides a quantized version of the interference to Tx2.

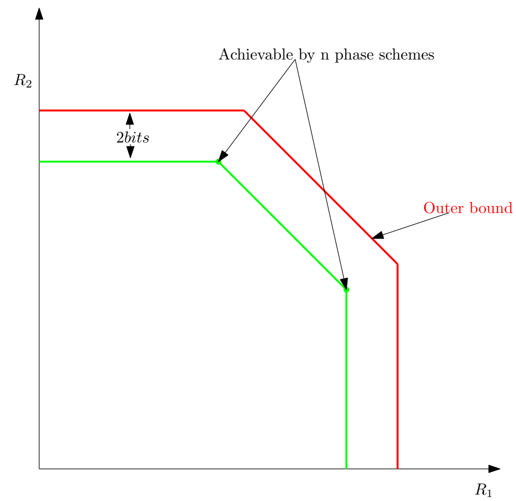

Figure 2. Illustration of bounds for capacity region for symmetric Gaussian IC. The corner points of the outer bound can be approximately achieved by our n-phase schemes. The same figure applies for the fading case, but with a larger capacity gap.

C. Extension to fading IC

For the fading IC with perfect feedback, we consider the same n-phase scheme, with the only difference that for phases $i > 1$, Tx2 transmits

$$X_2^{(i)N} = \frac{g_c^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+\mu_c}}. \tag{1}$$

We state the following result from [3] for fading models for using in our results for the fading IC.

Lemma 4 ([3])**.** *For an exponential random variable W with mean μ_W, and for any $\alpha, \beta \geq 0$,*

$$\begin{aligned} \log(\alpha + \beta\mu_W) &\geq \mathbf{E}\left[\log(\alpha + \beta W)\right] \\ &\geq \log(\alpha + \beta\mu_W) - \gamma \log(e) \end{aligned}$$

where γ is the Euler-Mascheroni constant. For a general distribution, if there exists a constant c such that $\mathbf{E}\left[\log\left(\frac{W}{\mu_W}\right)\right] \geq -c$, then

$$\begin{aligned} \log(\alpha + \beta\mu_W) &\geq \mathbf{E}\left[\log(\alpha + \beta W)\right] \\ &\geq \log(\alpha + \beta\mu_W) - c. \end{aligned}$$

It was shown in [3] that common fading models, such as Weibull and Gamma distribution satisfies the constraint in Lemma 4.

Now the following proposition shows that the simple extension of the scheme for static IC translates into a similar result for the fading IC.

Proposition 5. *The rate pair $(R_1, R_2) =$*

$$\left(\log(1 + \mu_d + \mu_c) - 2 - 3c, \mathbf{E}\left[\log^+\left[\frac{|g_d|^2}{1+\mu_c}\right]\right]\right)$$

is achievable by the scheme described in Subsection III-A, with the modification (1)*. The constant c is determined by the fading model according to Lemma 4.*

The proof for Proposition 5 relies on the idea that the penalty for treating the fading IC as a static IC with channel gains given by the mean of those in fading IC is bounded by the Jensen's gap for the logarithm of the (random) channel gain. In particular the Lemma 4 implies that for a class of widely used fading models, this penalty is bounded by a constant independent of the distribution of channel gains.

Theorem 6. *For the Rayleigh fading case, the two rate points achievable by n phase schemes, together with time sharing achieve the capacity region of symmetric feedback fading IC within $2 + 3\gamma \log(e) \approx 4.5$ bits, where γ is the Euler-Mascheroni constant. For an arbitrary fading model which satisfies the condition in Lemma 4 with a constant c, the gap is within $2 + 3c$ bits.*

Note that our computation of R_1 provides a closed form approximate expression for the 2-tap fading ISI channel capacity, described by $Y(t) = g_d(t)X(t) + g_c(t)X(t-1) + Z(t)$, as a by-product. This gives rise to the following corollary on the capacity of fading ISI channels.

Corollary 7. *If there exists c that satisfies the condition in Lemma 4 for g_d, g_c, then the capacity C_{F-ISI} of the 2-tap fading ISI channel is bounded by* $\log(1+\mu_d+\mu_c)+1 \geq C_{F-ISI} \geq \log(1+\mu_d+\mu_c)-1-2c$.

IV. Analysis

We describe modifications for the rate limited case with $C_{FB} \geq \log\left(1+|g_c|^2\right)$, $|g_c|^2 > 1$ and provide the analysis. The analysis carries over to the perfect feedback case by letting $C_{FB} \to \infty$. Later we analyze the fading case.

A. Static IC: rate limited case with $C_{FB} \geq \log\left(1+|g_c|^2\right)$, $|g_c|^2 > 1$

The only difference from scheme described in III, for the rate limited case is for user 2 in phases $i > 1$.

Instead of sending perfect feedback, the Rx2 performs Wyner-Ziv quantize-and-binning of the received symbols, considering that fact that Tx2 already has the side-information about its own transmitted symbols.

In phase i, Rx2 receives

$$Y_2^{(i)N} = g_d X_2^{(i)N} + g_c X_1^{(i)N} + Z_2^{(i)N}.$$

Using the Wyner-Ziv scheme with rate $\log\left(1+\frac{1+|g_c|^2}{D}\right)$, and distortion $D \geq 2$, Tx2 can recover $\hat{Y}_2^N$ over the feedback channel, provided that $C_{FB} \geq \log\left(1+|g_c|^2\right)$ and $|g_c|^2 > 1$.

After obtaining the quantized version $\hat{Y}_2^{(i)N} = g_d X_2^{(i)N} + g_c X_1^{(i)N} + Z_2^{(i)N} + Q_2^{(i)N}$ from the feedback, Tx2 strips out $g_d X_2^{(i)N}$ to obtain $g_c X_1^{(i)N} + Z_2^{(i)N} + Q_2^{(i)N}$and in the $(i+1)^{\text{th}}$ phase it sends:

$$X_2^{(i+1)N} = \frac{g_c X_1^{(i)N} + Z_2^{(i)N} + Q_2^{(i)N}}{\sqrt{1+D+|g_c|^2}}$$

Also note that Rx2 has knowledge of the quantization noise $Q_2^{(i)N}$ which it can use in the noise refinement. The transmission scheme is illustrated in Table II. Note that the scheme reduces to perfect feedback case by letting $D \to 0$.

Table II
$n-$PHASE SCHEME FOR RATE LIMITED FEEDBACK CASE

User	Phase 1	Phase 2	.	.	Phase n
1	$X_1^{(1)N}$	$X_1^{(2)N}$	.	.	$X_1^{(n)N}$
2	$X_2^{(1)N}$	$\frac{g_c X_1^{(1)N} + Z_2^{(1)N} + Q_2^{(1)N}}{\sqrt{1+D+\lvert g_c\rvert^2}}$	.	.	$\frac{g_c X_1^{(n-1)N} + Z_2^{(n-1)N} + Q_2^{(n-1)N}}{\sqrt{1+D+\lvert g_c\rvert^2}}$

1) Decoding at Rx1: At the end of n blocks Rx1 collects $\mathbf{Y_1}^N = \left(Y_1^{(1)N}, \ldots Y_1^{(n)N}\right)$ and decodes W_1 such that $\left(\mathbf{X_1}^N(W_1), \mathbf{Y_1}^N\right)$ is jointly typical, where $\mathbf{X_1}^N = \left(X_1^{(1)N}, \ldots X_1^{(n)N}\right)$. Using standard techniques it follows that for the n-phase scheme as $N \to \infty$ user 1 can achieve the rate $\frac{1}{n}\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)$ where $|K_{\mathbf{Y_1}}(n)|$ denotes the determinant of covariance matrix for the n-phase scheme. Letting $n \to \infty$, Rx1 can achieve the rate $\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)$. We need to evaluate $\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)$ where $|K_{\mathbf{Y_1}}(n)|$ denotes the determinant of covariance matrix for the n-phase scheme. The following lemma is helpful in evaluating this limit.

Lemma 8. *If* $A_1 = [|a|]$, $A_2 = \begin{bmatrix} |a| & b \\ b^* & |a| \end{bmatrix}$, $A_3 = \begin{bmatrix} |a| & b & 0 \\ b^* & |a| & b \\ 0 & b^* & |a| \end{bmatrix}$, $A_4 = \begin{bmatrix} |a| & b & 0 & 0 \\ b^* & |a| & b & 0 \\ 0 & b^* & |a| & b \\ 0 & 0 & b^* & |a| \end{bmatrix}$ *etc. with* $|a|^2 > 4|b|^2$, *then*

$$\liminf_{n\to\infty}\frac{1}{n}\log(|A_n|) \geq \log\left(\frac{|a|}{2}\right).$$

Proof: The proof is given in Appendix A. ∎

For the n-phase scheme, the $K_{\mathbf{Y_1}}(n)$ matrix has the form A_{n+1}, as defined in Lemma 8 after identifying $|a| = 1+|g_c|^2+|g_d|^2$ and $b = \frac{g_d g_c^2}{\sqrt{1+D+|g_c|^2}}$. Also, $K_{\mathbf{Y_1}|\mathbf{X_1}}(n)$ is a diagonal matrix with first diagonal element $1+|g_c|^2$ and rest of the diagonal elements $1+|g_c|^2\frac{(1+D)}{1+|g_c|^2+D}$. Hence $\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right| = \left(1+|g_c|^2\frac{(1+D)}{1+|g_c|^2+D}\right)^n\left(1+|g_c|^2\right)$. Since $\left(1+|g_c|^2\frac{(1+D)}{1+|g_c|^2+D}\right)^n \leq (2+D)^n$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|\right) \leq \log\left(2+D\right).$$

Therefore the following rate is achievable for user 1

$$\begin{aligned}R_1 &\leq \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|}\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\log\left(|K_{\mathbf{Y_1}}(n)|\right)-\log\left(2+D\right)\end{aligned}$$

We have

$$\liminf_{n\to\infty}\frac{1}{n}\log\left(|K_{\mathbf{Y_1}}(n)|\right) \geq \log\left(\frac{1+|g_c|^2+|g_d|^2}{2}\right)$$

using Lemma 8. Hence

$$R_1 \leq \log\left(1+|g_c|^2+|g_d|^2\right)-\log\left(2+D\right)-1$$

is achievable. Using $D \leq 2$ if follows that

$$R_1 \leq \log\left(1+|g_c|^2+|g_d|^2\right)-3$$

is achievable for the rate limited case. For perfect feedback case

$$R_1 \leq \log\left(1+|g_c|^2+|g_d|^2\right)-2$$

easily follows by setting $D=0$.

2) Decoding at Rx2: For user 2 we can use a block variant of Schalkwijk-Kailath scheme [2] to achieve $R_2 = \log^+\left(\frac{|g_d|^2}{1+|g_c|^2}\right)$. The key idea is that the interference-plus-noise sent in subsequent slots can indeed refine the symbols of the previous slot. The chain of refinement over n phases compensate for the fact that the information symbols are sent only in the first phase. Rx2 can remove the quantization noise $Q_2^{(i)N}$ from its recieved symbols for $i>1$

$$Y_2^{(i)N} = g_d\left(\frac{g_cX_1^{(i)N}+Z_2^{(i)N}+Q_2^{(i)N}}{\sqrt{1+D+|g_c|^2}}\right)+g_cX_1^{(i)N}+Z_2^{(i)N}$$

to obtain

$$Y_2^{'(i)N} = g_d\left(\frac{g_cX_1^{(i)N}+Z_2^{(i)N}}{\sqrt{1+D+|g_c|^2}}\right)+g_cX_1^{(i)N}+Z_2^{(i)N}.$$

for $i>1$ and let $Y_2^{'(1)N}=Y_2^{(1)N}$. Now

$$\begin{aligned}\tilde{Y}_2^N &= \sum_{i=1}^{n}\left(\frac{-g_d}{\sqrt{1+D+|g_c|^2}}\right)^{n-i}Y_2^{'(i)N}\\ &= g_d\left(\frac{-g_d}{\sqrt{1+D+|g_c|^2}}\right)^{n-1}X_2+g_cX_1^{(n)N}+Z_2^{(n)N}.\end{aligned}$$

Now Rx2 decodes for its message from $\tilde{Y}_2^N$. Hence Rx2 can obtain the rate

$$\begin{aligned}R_2 &\leq \liminf_{n\to\infty}\frac{1}{n}\log\left(1+\left(\frac{|g_d|^2}{1+D+|g_c|^2}\right)^{n-1}\frac{|g_d|^2}{1+|g_c|^2}\right)\\ &= \left(\log\left(\frac{|g_d|^2}{1+D+|g_c|^2}\right)\right)^+.\end{aligned}$$

Since $D\leq 2$, $R_2=\log^+\left(\frac{|g_d|^2}{3+|g_c|^2}\right)$ is achievable for rate-limited case and by setting $D=0$, $R_2=\log^+\left(\frac{|g_d|^2}{1+|g_c|^2}\right)$ is achievable for perfect feedback case.

B. Sketch of Analysis for fading case

Similar to static case it follows that User 1 achieves the following rate with jointly typical decoding

$$R_1 \leq \lim_{n\to\infty} \frac{1}{n}\mathbf{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|}\right)\right]$$

where where $K_{\mathbf{Y_1}}(n)$ is the covariance matrix for the n-phase scheme similar to the static case, but it now contains random elements. In Appendix D, we show that this limit is at least $\log(1+\mu_d+\mu_c)-2-3c$. The proof relies on Lemma 8 and Lemma 4.

Rx2 can perform noise refinement similar to the non-fading case and achieve $R_2 \leq \mathbf{E}\left[\log^+\left(\frac{|g_d|^2}{1+\mu_c}\right)\right]$.

C. Capacity gap for fading IC

We can obtain the following outer bounds for i.i.d. fading Gaussian interference channels with feedback from [3]:

$$\begin{aligned}
R_1, R_2 &\leq \mathbf{E}\left[\log\left(|g_d|^2+|g_c|^2+1\right)\right] \\
R_1+R_2 &\leq \mathbf{E}\left[\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\right] \\
&\quad + \mathbf{E}\left[\log\left(|g_d|^2+|g_c|^2+2\,|g_d|\,|g_c|+1\right)\right].
\end{aligned}$$

The outer bounds again reduce to a pentagonal region with two non-trivial corner points (see Figure 2). Our n-phase scheme can achieve the two corner points within $2+3c$ bits for each user. The computations can be done similar to the static case as in Appendix C, and using Lemma 4.

V. Conclusion

In this paper we proposed a scheme for the interference channel with feedback which did not use rate-splitting and enabled the use of point-to-point codes. We showed the approximate optimality of this scheme for static as well as a class of fading channels. Part of the ongoing work is to achieve the approximate capacity without rate splitting, when feedback capacity is less than required by the current scheme. Another line of ongoing work is to demonstrate the (approximate) optimality of the scheme in a rateless setting.

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APPENDIX A
MATRIX DETERMINANT: ASYMPTOTIC BEHAVIOR

The following recursion easily follows:

$$|A_n| = |a|\,|A_{n-1}| - |b|^2\,|A_{n-2}|$$

with $|A_1| = |a|\,, |A_2| = |a|^2 - |b|^2$. Also $|A_0|$ can be consistently defined to be 1. The characteristic equation for this recursive relation is given by: $\lambda^2 - |a|\,\lambda + |b|^2 = 0$ and the characteristic roots are given by:

$$\lambda_1 = \frac{|a| + \sqrt{|a|^2 - 4\,|b|^2}}{2}, \lambda_2 = \frac{|a| - \sqrt{|a|^2 - 4\,|b|^2}}{2}.$$

Now the solution of the recursive system is given by:

$$|A_n| = c_1\lambda_1^n + c_2\lambda_2^n$$

with the boundary conditions

$$\begin{aligned} 1 &= c_1 + c_2 \\ |a| &= c_1\lambda_1 + c_2\lambda_2. \end{aligned}$$

It can be easily seen that $c_1 > 0$, $\lambda_1 > \lambda_2 > 0$ since $|a|^2 > 4\,|b|^2$. Now

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log\left(|A_n|\right) &= \lim_{n\to\infty} \frac{1}{n} \log\left(c_1\lambda_1^n + c_2\lambda_2^n\right) \\ &= \lim_{n\to\infty} \frac{1}{n}\left(\log\left(\lambda_1^n\right) + \log\left(c_1 + c_2\frac{\lambda_2^n}{\lambda_1^n}\right)\right) \\ &\overset{(a)}{=} \log\left(\lambda_1\right) \\ &= \log\left(\frac{|a| + \sqrt{|a|^2 - 4\,|b|^2}}{2}\right). \end{aligned}$$

The step (a) follows because $1 > \frac{\lambda_2}{\lambda_1} > 0$ and $c_1 > 0$.

APPENDIX B
IMPLEMENTATION USING OFDM TECHNIQUES

With our scheme the channel for user 1 can be to an inter-symbol interference channel since user 2 amplifies and forwards the interference-plus-noise. We modify our scheme slightly to be able to use the OFDM techniques. We use $n+1$ phase and user 1 repeats the symbols from phase 1 on the phase $n+1$. The scheme is described in Table III. A succinct notation FB: $X_1^{(1)N}$ is used to indicate that the interference of $X_1^{(1)N}$ and the noise is amplified and forward similar to Table I.

User	Phase 1	Phase 2	.	Phase n	Phase $n+1$
1	$X_1^{(1)N}$	$X_1^{(2)N}$	.	$X_1^{(n)N}$	$X_1^{(1)N}$
2	$X_2^{(1)N}$	FB: $X_1^{(1)N}$	.	FB: $X_1^{(n-1)N}$	FB: $X_1^{(n)N}$

Table III
MODIFIED N-PHASE SCHEME FOR OFDM IMPLEMENTATION

Ignoring the first phase at Rx1 we get the following for the first user for the phases 2 to $n+1$

$$\mathbf{Y_1} = \begin{bmatrix} d & f & 0 & 0 & . & . & 0 \\ 0 & d & f & 0 & . & . & . \\ 0 & 0 & d & f & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & d & f \\ f & 0 & . & . & . & 0 & d \end{bmatrix} \mathbf{X_1} + \mathbf{Z_1}$$

where $\mathbf{Y_1^N} = transpose\left(Y_1^{(2)N}, \ldots, Y_1^{(n+1)N}\right), \mathbf{X_1^N} = transpose\left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$ and $f = g_d$ and $d = \frac{g_c^2}{\sqrt{1+|g_c|^2}}$ and the noise power in each phase being $1 + \frac{|g_c|^2}{1+|g_c|^2} = \frac{1+2|g_c|^2}{1+|g_c|^2}$. Since the transfer matrix is circulant, it can be diagonalized using the DFT matrices and the n eigenvalues are given by

$$\lambda_j = d + f\exp\left(\frac{2\pi i j}{n}\right), \quad j = 0, \ldots, n-1$$

where i is the imaginary unit. This yields n point to point channels with $SNR_j = \frac{\left|d+f\exp\left(\frac{2\pi ij}{n}\right)\right|^2}{\frac{1+2|g_c|^2}{1+|g_c|^2}}$. Hence the rate for the user 1 is given by:

$$\begin{aligned}R_1 &\leq \frac{1}{n+1}\sum_{j=0}^{n-1}\log\left(1+\frac{\left|d+f\exp\left(\frac{2\pi ij}{n}\right)\right|^2}{\frac{1+2|g_c|^2}{1+|g_c|^2}}\right)\\ &= \frac{1}{n+1}\sum_{j=0}^{n-1}\log\left(1+\frac{|d|^2+|f|^2+2\,|df|\cos\left(\phi+\frac{2\pi j}{n}\right)}{\frac{1+2|g_c|^2}{1+|g_c|^2}}\right)\end{aligned}$$

where ϕ is to account for any phase difference between the complex numbers f and d. As as $n\to\infty$ we would get a Riemann integral. We get

$$R_1 \leq \frac{1}{2\pi}\int_0^{2\pi}\log\left(p+q\cos\left(\phi+\theta\right)\right)d\theta$$

where $p = 1+\frac{|d|^2+|f|^2}{\frac{1+2|g_c|^2}{1+|g_c|^2}}$ and $q = \frac{2|df|}{\frac{1+2|g_c|^2}{1+|g_c|^2}}$. Now we use the following fact and periodicity of $\cos(.)$ function.

Fact 9. *For $p \geq q$, $\frac{1}{2\pi}\int_0^{2\pi}\log\left(p+q\cos\left(\theta\right)\right)d\theta = \log\left(\frac{p+\sqrt{p^2-q^2}}{2}\right)$*

Using the previous fact it easily follows that $R_1 \leq \log\left(\frac{p}{2}\right) = \log\left(1+\frac{|d|^2+|f|^2}{\frac{1+2|g_c|^2}{1+|g_c|^2}}\right)-1$ is achievable. Now

$$|d|^2+|f|^2+1+\frac{1+2\,|g_c|^2}{1+|g_c|^2} = 1+|g_c|^2+|g_d|^2$$

and hence

$$R_1 \leq \log\left(1+|g_c|^2+|g_d|^2\right)-2$$

achievablity follows.

APPENDIX C
APPROXIMATE CAPACITY USING N PHASE SCHEMES

The following outer bounds follow from Suh-Tse.

$$\begin{aligned}R_1 &\leq \log\left(1+|g_c|^2+|g_d|^2\right)\\ R_1+R_2 &\leq \log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\\ &\quad+\log\left(1+|g_d|^2+|g_c|^2+2\,|g_d|\,|g_c|\right)\\ R_2 &\leq \log\left(1+|g_c|^2+|g_d|^2\right).\end{aligned}$$

The above outer-bound region is a polytope with the following two non-trivial corner points:

$$\left\{\begin{array}{ccc} R_1 &=& \log\left(1+|g_c|^2+|g_d|^2\right)\\ R_2 &=& \log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)+\log\left(1+\frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\end{array}\right\}$$

$$\left\{\begin{array}{ccc} R_1 &=& \log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)+\log\left(1+\frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\\ R_2 &=& \log\left(1+|g_c|^2+|g_d|^2\right)\end{array}\right\}$$

We can achieve these rate points within 2 bits for each user using the n-phase schemes. The only important point to verify is in the following claim

Claim 10. $\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)+\log\left(1+\frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)-\left(\log\left(\frac{|g_d|^2}{1+|g_c|^2}\right)\right)^+ \leq 2$

Proof: We have $\frac{2|g_d||g_c|}{|g_d|^2+|g_c|^2} \leq 1$ due to AM-GM inequality. Hence

$$\log\left(1+\frac{2\left|g_d\right|\left|g_c\right|}{1+\left|g_d\right|^2+\left|g_c\right|^2}\right) \leq 1.$$

Now if $\left(\log\left(\frac{|g_d|^2}{1+|g_c|^2}\right)\right)^+ = 0$ then $\frac{|g_d|^2}{1+|g_c|^2} \leq 1$ and hence $\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right) \leq \log(2) = 1$. If $\left(\log\left(\frac{|g_d|^2}{1+|g_c|^2}\right)\right)^+ > 0$ then $\frac{|g_d|^2}{1+|g_c|^2} > 1$ and hence again $\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right) - \left(\log\left(\frac{|g_d|^2}{1+|g_c|^2}\right)\right)^+ = \log\left(1+\frac{1+|g_c|^2}{|g_d|^2}\right) < 1$. ■

Appendix D
Analysis of n phase scheme for Fading IC with feedback

User 1 can achieve the following rate.

$$R_1 \leq \lim_{n\to\infty}\frac{1}{n}\mathbf{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|}\right)\right].$$

It can be easily verified that:

$$\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\left[\log\left(\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|\right)\right] = \mathbf{E}\left[\log\left(1+\frac{\left|g_c'\right|^2}{1+\left|g_c\right|^2}\right)\right]$$

where g_c and g_c' are independent with same fading distribution. Indeed it is also true that:

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|\right) = \mathbf{E}\left[\log\left(1+\frac{\left|g_c'\right|^2}{1+\left|g_c\right|^2}\right)\right]$$

using law of large numbers. Now

$$\begin{aligned}\mathbf{E}\left[\log\left(1+\frac{\left|g_c'\right|^2}{1+\left|g_c\right|^2}\right)\right] &= \mathbf{E}\left[\log\left(1+\left|g_c\right|^2+\left|g_c'\right|^2\right)\right] \\ &\quad - \mathbf{E}\left[\log\left(1+\left|g_c'\right|^2\right)\right] \\ &\leq \log\left(1+2\mathbf{E}\left[\left|g_c\right|^2\right]\right) \\ &\quad - \log\left(1+\mathbf{E}\left[\left|g_c\right|^2\right]\right) + c \\ &\leq 1+c\end{aligned}$$

where we used the Jensen's inequality and Lemma 4 .

Lemma 11.

$$\begin{aligned}\mathit{lim}_{n\to\infty}\frac{1}{n}\mathbf{E}\left[\log\left(\left|K_{\mathbf{Y_1}}(n)\right|\right)\right] \geq \mathit{lim}_{n\to\infty}\frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) \\ - 2c\end{aligned}$$

where $\hat{K}_{\mathbf{Y_1}}(n)$ has all the $g_c(i)$'s replaced by $\sqrt{\mathbf{E}\left[\left|g_c\right|^2\right]}$ and similarly for $g_d(i)$'s.

Proof: The proof involves expanding the matrix determinant and repeated application of Lemma 4. The details are given in Appendix E. ■

Hence it follows that:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\left[\log\left(\left|K_{\mathbf{Y_1}}(n)\right|\right)\right] \geq \log\left(1+\mathbf{E}\left[\left|g_d\right|^2\right]+\mathbf{E}\left[\left|g_c\right|^2\right]\right) \\ -(1+2c)\end{aligned}$$

using the result for the non-fading case and hence

$$R_1 \leq \log\left(1+\mathbf{E}\left[\left|g_d\right|^2\right]+\mathbf{E}\left[\left|g_c\right|^2\right]\right) - (2+3c)$$

is achievable. Also for the Rayleigh fading case $c = \gamma\log(e)$ and so $(2+3\gamma\log(e)) = 4.4982$ numerically.

Appendix E
Fading matrix

$$\begin{aligned}
&\mathbf{E}\left[\log\left(\left|K_{\mathbf{Y_1}}(n)\right|\right)\right] \\
&\quad = \mathbf{E}\left[\log\left(\left(1+\left|g_c(n)\right|^2+\left|g_d(n)\right|^2\right)\left|K_{\mathbf{Y_1}}(n-1)\right| - \frac{\left|g_d(n)\right|^2\left|g_c(n)\right|^2\left|g_c(n-1)\right|^2}{1+\mathbf{E}\left[\left|g_c\right|^2\right]}\left|K_{\mathbf{Y_1}}(n-2)\right|\right)\right] \\
&\quad \geq \mathbf{E}\left[\log\left(\left(1+\mathbf{E}\left[\left|g_c\right|^2\right]+\mathbf{E}\left[\left|g_d\right|^2\right]\right)\left|K_{\mathbf{Y_1}}(n-1)\right| - \frac{\mathbf{E}\left[\left|g_d\right|^2\right]\mathbf{E}\left[\left|g_c\right|^2\right]\left|g_c(n-1)\right|^2}{1+\mathbf{E}\left[\left|g_c\right|^2\right]}\left|K_{\mathbf{Y_1}}(n-2)\right|\right)\right] - 2c.
\end{aligned}$$

The first step is by expanding the determinant. We use Lemma 4 twice in the second step. This is justified because the coefficients of $\left|g_c(n)\right|^2$ and $\left|g_d(n)\right|^2$in the above expansion are non-negative (due to the fact that all the matrices involved are covariance matrices), and the coefficients themselves are independent of $\left|g_c(n)\right|^2$ and $\left|g_d(n)\right|^2$. This procedure can be carried out n times and it follows that:

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\left[\log\left(\left|K_{\mathbf{Y_1}}(n)\right|\right)\right] \geq \lim_{n\to\infty}\frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) \\
- 2c
\end{aligned}$$

where $\hat{K}_{\mathbf{Y_1}}(n)$ has all the $g_c(i)$'s replaced by $\sqrt{\mathbf{E}\left[\left|g_c\right|^2\right]}$ and similarly for $g_d(i)$'s.